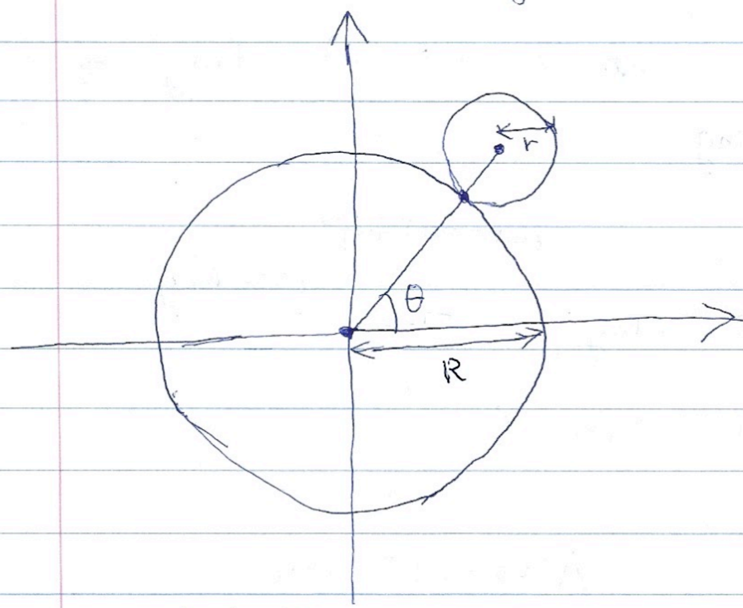
Goldstein

2.14. We use the following coordinate system.




We adopt the same Lagrangian as 2.13 with some modification.

1. radius of circle change from $R \to R+r$.

2. Additional kinetic energy for relative motion of the constituents of the small hoop about its center of mass (i.e. rotational kinetic energy).

We let $R = R+r$ for simplicity of notation (This is simply a redefinition of $R$). We begin by treating $R$ as a variable, the distance from the COM of the hoop to the origin, then apply a constraint via Lagrange multiplier.

$T = \frac{1}{2}m[R^2\dot{\theta}^2 + \dot{R}^2] + \frac{1}{2}mr^2\dot{\theta}^2$

$V = mgR\sin\theta$

$L = \frac{1}{2}m[R^2\dot{\theta}^2 + \dot{R}^2 + r^2\dot{\theta}^2] - mgR\sin\theta$

$\frac{dL}{dR} = mR\dot{\theta}^2 - mg\sin\theta$, $\frac{\partial L}{\partial \dot{R}} = m\dot{R}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\ddot{R}$

$\frac{dL}{d\theta} = -mgR\cos\theta$, $\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} + mr^2\dot{\theta}$,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} + mr^2\ddot{\theta}$

$= 2mR\dot{R}\dot{\theta} + m\ddot{\theta}(R^2 + r^2)$.

Apply constraint that $R = a$, $\dot{R} = \ddot{R} = 0$.

Eqm: $m\ddot{R} - m\dot{\theta}^2R + mg\sin\theta = 0$

$2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} + r^2\ddot{\theta}m + mgR\cos\theta = 0$

$\Downarrow$

$-m\dot{\theta}^2 a + mg\sin\theta = \lambda$

$\cancel{mR}\ ma^2\ddot{\theta} + a^2\ddot{\theta}m + mga\cos\theta = 0$.

$\frac{d\lambda}{dt} = -2m\dot{\theta}\ddot{\theta}a + mg\cos\theta\dot{\theta} = \frac{d\lambda}{d\theta}\dot{\theta}$

$\Rightarrow \frac{d\lambda}{dt} = -2m\ddot{\theta}a + mg\cos\theta$.

$2ma^2\ddot{\theta} = -mga\cos\theta \Rightarrow 2ma\ddot{\theta} = -mg\cos\theta$.

$\frac{d\lambda}{d\theta} = 2mg\cos\theta$.

$\lambda = 2mg\sin\theta + c$.

Boundary condition dictates $C = -mg$.

We thus obtain

$$\lambda(\theta) = Q(\theta) = 2mg \sin\theta - mg$$

Setting it equal to find the critical point of the force of constraint:

$$2mg \sin\theta - mg = 0.$$

$$2mg \sin\theta = mg$$

$$\sin\theta = \frac{1}{2}$$

$$\boxed{\theta^* = \sin^{-1}\left(\frac{1}{2}\right)}$$

Davidson Cheng
12.28.2023